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A Brooks type theorem for the maximum local edge connectivity

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Abstract

For a graph G, let $\chi(G)$ and $\lambda(G)$ denote the chromatic number of G and the maximum local edge connectivity of G, respectively. A result of Dirac [4] implies that every graph G satisfies $\chi(G) \leq \lambda(G)+1$. In this paper we characterize the graphs G for which $\chi(G) = \lambda(G)+1$. The case $\lambda(G) = 3$ was already solved by Alboulker *et al.* [1]. We show that a graph G with $\lambda(G) = k \geq 4$ satisfies $\chi(G) = k+1$ if and only if G contains a block which can be obtained from copies of K_{k+1} by repeated applications of the Hajós join.

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1 Introduction and main result

The paper deals with the classical vertex coloring problem for graphs. The term graph refers to a finite undirected graph without loops and without multiple edges. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of G such that each vertex receives a color and adjacent vertices receive different colors. There

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are several degree bounds for the chromatic number. For a graph G, let $\delta(G) = \min_{v\in V(G)} d_G(v)$ and $\Delta(G) = \max_{v\in V(G)} d_G(v)$ denote the *minimum degree* and the *maximum degree* of G, respectively. Furthermore, let

$$\mathrm{col}(G) = 1 + \max_{H\subseteq G} \delta(H)$$

denote the *coloring number* of G, and let

$$\mathrm{mad}(G) = \max_{\varnothing\neq H\subseteq G} \frac{2|E(H)|}{|V(H)|}$$

denote the *maximum average degree* of G. By $H \subseteq G$ we mean that H is a subgraph of G. If G is the *empty graph*, that is, $V(G) = \varnothing$, we briefly write $G = \varnothing$ and define $\delta(G) = \Delta(G) = \mathrm{mad}(G) = 0$ and $\mathrm{col}(G) = 1$. A simple sequential coloring argument shows that $\chi(G) \leq \mathrm{col}(G)$, which implies that every graph G satisfies

$$\chi(G) \leq \mathrm{col}(G) \leq \lfloor \mathrm{mad}(G) \rfloor + 1 \leq \Delta(G) + 1.$$

These inequalities were discussed in a paper by Jensen and Toft [10]. Brooks' famous theorem provides a characterization for the class of graphs G satisfying $\chi(G) = \Delta(G) + 1$. Let $k \geq 0$ be an integer. For $k \neq 2$, let $\mathcal{B}_k$ denote the class of complete graphs having order $k + 1$; and let $\mathcal{B}_2$ denote the class of odd cycles. A graph in $\mathcal{B}_k$ has maximum degree k and chromatic number $k + 1$. Brooks' theorem [2] is as follows.

Theorem 1.1 (Brooks 1941) *Let G be a non-empty graph. Then $\chi(G) \leq \Delta(G) + 1$ and equality holds if and only if G has a connected component belonging to the class $\mathcal{B}_{\Delta(G)}$.*

In this paper we are interested in connectivity parameters of graphs. Let G be a graph with at least two vertices. The *local connectivity* $\kappa_G(v, w)$ of distinct vertices v and w is the maximum number of internally vertex disjoint v-w paths of G. The *local edge connectivity* $\lambda_G(v, w)$ of distinct vertices v and w is the maximum number of edge-disjoint v-w paths of G. The *maximum local connectivity* of G is

$$\kappa(G) = \max\{\kappa_G(v, w) \mid v, w \in V(G), v \neq w\},$$

and the *maximum local edge connectivity* of G is

$$\lambda(G) = \max\{\lambda_G(v, w) \mid v, w \in V(G), v \neq w\}.$$

For a graph G having only one vertex, we define $\kappa(G) = \lambda(G) = 0$. Clearly, the definition implies that $\kappa(G) \leq \lambda(G)$ for every graph G. By a result of Mader [11] it follows that $\delta(G) \leq \kappa(G)$. Since κ is a monotone graph parameter in the sense that $H \subseteq G$ implies $\kappa(H) \leq \kappa(G)$, it follows that every graph G satisfies $\mathrm{col}(G) \leq \kappa(G) + 1$. Consequently, every graph G satisfies

$$\chi(G) \leq \mathrm{col}(G) \leq \kappa(G) + 1 \leq \lambda(G) + 1 \leq \Delta(G) + 1. \tag{1.1}$$

Our aim is to characterize the class of graphs G for which $\chi(G) = \lambda(G) + 1$. For such a characterization we use the fact that if we have an optimal coloring of each block of a graph G, then we can combine these colorings to an optimal coloring of G by permuting colors in the blocks if necessary. For every non-empty graph G, we thus have

$$\chi(G) = \max\{\chi(H) \mid H \text{ is a block of } G\}. \tag{1.2}$$

We also need a famous construction, first used by Hajós [9]. Let G_1 and G_2 be two vertex-disjoint graphs and, for $i = 1, 2$, let $e_i = v_i w_i$ be an edge of G_i. Let G be the graph obtained from G_1 and G_2 by deleting the edges e_1 and e_2 from G_1 and G_2, respectively, identifying the vertices v_1 and v_2, and adding the new edge $w_1 w_2$. We then say that G is the *Hajós join* of G_1 and G_2 and write $G = (G_1, v_1, w_1) \bigtriangleup (G_2, v_2, w_2)$ or briefly $G = G_1 \bigtriangleup G_2$.

For an integer $k \geq 0$ we define a class $\mathcal{H}_k$ of graphs as follows. If $k \leq 2$, then $\mathcal{H}_k = \mathcal{B}_k$. The class $\mathcal{H}_3$ is the smallest class of graphs that contains all odd wheels and is closed under taking Hajós joins. Recall that an *odd wheel* is a graph obtained from on odd cycle by adding a new vertex and joining this vertex to all vertices of the cycle. If $k \geq 4$, then $\mathcal{H}_k$ is the smallest class of graphs that contains all complete graphs of order $k+1$ and is closed under taking Hajós joins. Our main result is the following counterpart of Brooks' theorem. In fact, Brooks' theorem may easily be deduced from it.

Theorem 1.2 *Let G be a non-empty graph. Then $\chi(G) \leq \lambda(G) + 1$ and equality holds if and only if G has a block belonging to the class $\mathcal{H}_{\lambda(G)}$.*

For the proof of this result, let G be a non-empty graph with $\lambda(G) = k$. By (1.1), we obtain $\chi(G) \leq k + 1$. By an observation of Hajós [9] it follows that every graph in $\mathcal{H}_k$ has chromatic number $k + 1$. Hence if some block of G belongs to $\mathcal{H}_k$, then (1.2) implies that $\chi(G) = k+1$. So it only remains to show that if $\chi(G) = k+1$, then some block of G belongs to $\mathcal{H}_k$. For proving this, we shall use the critical graph method, see [12].

A graph G is *critical* if every proper subgraph H of G satisfies $\chi(H) < \chi(G)$. We shall use the following two properties of critical graphs. As an immediate consequence of (1.2) we obtain that if G is a critical graph, then $G = \varnothing$ or G contains no separating vertex, implying that G is its only block. Furthermore, every graph contains a critical subgraph with the same chromatic number.

Let G be a non-empty graph with $\lambda(G) = k$ and $\chi(G) = k+1$. Then G contains a critical subgraph H with chromatic number $k+1$, and we obtain that $\lambda(H) \leq \lambda(G) = k$. So the proof of Theorem 1.2 is complete if we can show that H is a block of G which belongs to $\mathcal{H}_k$. For an integer $k \geq 0$, let $\mathcal{C}_k$ denote the class of graphs H such that H is a critical graph with chromatic number $k+1$ and with $\lambda(H) \leq k$. We shall prove that the two classes $\mathcal{C}_k$ and $\mathcal{H}_k$ are the same.

2 Connectivity of critical graphs

In this section we shall review known results about the structure of critical graphs. First we need some notation. Let G be an arbitrary graph. For an integer $k \geq 0$, let $\mathcal{CO}_k(G)$ denote the set of all colorings of G with color set $\{1, 2, \ldots, k\}$. Then a function $f : V(G) \to \{1, 2, \ldots, k\}$ belongs to $\mathcal{CO}_k(G)$ if and only if $f^{-1}(c)$ is an independent vertex set of G (possibly empty) for every color $c \in \{1, 2, \ldots, k\}$. A set $S \subseteq V(G) \cup E(G)$ is called a *separating set* of G if $G - S$ has more components than G. A vertex v of G is called a *separating vertex* of G if $\{v\}$ is a separating set of G. An edge e of G is called a *bridge* of G if $\{e\}$ is a separating set of G. For a vertex set $X \subseteq V(G)$, let $\partial_G(X)$ denote the set of all edges of G having exactly one end in X. Clearly, if G is connected and $\varnothing \neq X \subsetneq V(G)$, then $F = \partial_G(X)$ is a separating set of edges of G. The converse is not true. However if F is a minimal separating edge set of a connected graph G, then $F = \partial_G(X)$ for some vertex set X. As a consequence of Menger's theorem about edge connectivity, we obtain that if v and w are two distinct vertices of G, then

$$\lambda_G(v, w) = \min\{|\partial_G(X)| \mid X \subseteq V(G), v \in X, w \notin X\}.$$

Color critical graphs were first introduced and investigated by Dirac in the 1950s. He established the basic properties of critical graphs in a series of papers [3], [4] and [5]. Some of these basic properties are listed in the next theorem.

Theorem 2.1 (Dirac 1952) *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geq 0$. Then the following statements hold:*

(a) $\delta(G) \leq k$

(b) *If $k = 0, 1$, then G is a complete graph of order $k+1$; and if $k = 2$, then G is an odd cycle.*

(c) *No separating vertex set of G is a clique of G. As a consequence, G is connected and has no separating vertex, i.e., G is a block.*

(d) *If v and w are two distinct vertices of G, then $\lambda_G(v, w) \geq k$. As a consequence G is k-edge-connected.*

Theorem 2.1(a) leads to a very natural way of classifying the vertices of a critical graph into two classes. Let G be a critical graph with chromatic number $k+1$. The vertices of G having degree k in G are called *low vertices* of G, and the remaining vertices are called *high vertices* of G. So any high vertex of G has degree at least $k+1$ in G. Furthermore, let G_L be the subgraph of G induced by the low vertices of G, and let G_H be the subgraph of G induced by the high vertices of G. We call G_L the *low vertex subgraph* of G and G_H the *high vertex subgraph* of G. This classification is due to Gallai [8] who proved the following theorem. Note that statements (b) and (c) of Gallai's theorem are simple consequences of statement (a), which is an extension of Brooks' theorem.

Theorem 2.2 (Gallai 1963) *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geq 1$. Then the following statements hold:*

(a) *Every block of G_L is a complete graph or an odd cycle*

(b) *If $G_H = \varnothing$, then G is a complete graph of order $k+1$ if $k \neq 2$, and G is an odd cycle if $k = 2$.*

(c) *If $|V(G_H)| = 1$, then either G has a separating vertex set of two vertices or $k = 3$ and G is an odd wheel.*

As observed by Dirac, a critical graph is connected and contains no separating vertex. Dirac [3] and Gallai [8] characterized critical graphs having a separating vertex set of size two. In particular, they proved the following theorem, which shows how to decompose a critical graph having a separating vertex set of size two into smaller critical graphs.

Theorem 2.3 (Dirac 1952 and Gallai 1963) *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geq 3$, and let $S \subseteq V(G)$ be a separating vertex set of G with $|S| \leq 2$. Then S is an independent vertex set of G consisting of two vertices, say v and w, and $G - S$ has exactly two components H_1 and H_2. Moreover, if $G_i = G[V(H_i) \cup S]$ for $i = 1, 2$, we can adjust the notation so that for some coloring $f_1 \in \mathcal{CO}_k(G_1)$ we have $f_1(v) = f_1(w)$. Then the following statements hold:*

(a) *Every coloring $f \in \mathcal{CO}_k(G_1)$ satisfies $f(v) = f(w)$ and every coloring $f \in \mathcal{CO}_k(G_2)$ satisfies $f(v) \neq f(w)$.*

(b) *The subgraph $G_1' = G_1 + vw$ obtained from G_1 by adding the edge vw is critical and has chromatic number $k+1$.*

(c) *The vertices v and w have no common neighbor in G_2 and the subgraph $G_2' = G_2/S$ obtained from G_2 by identifying v and w is critical and has chromatic number $k+1$.*

Dirac [6] and Gallai [8] also proved the converse theorem, that G is critical and has chromatic number $k+1$ provided that G_1' is critical and has chromatic number $k+1$ and G_2 obtained from the critical graph G_2' with chromatic number $k+1$ by splitting a vertex into v and w has chromatic number k.

Hajós [9] invented his construction to characterize the class of graphs with chromatic number at least $k+1$. Another advantage of the Hajós join is the well known fact that it not only preserve the chromatic number, but also criticality. It may be viewed as a special case of the Dirac–Gallai construction, described above.

Theorem 2.4 (Hajós 1961) *Let $G = G_1 \bigtriangleup G_2$ be the Hajós join of two graphs G_1 and G_2, and let $k \geq 3$ be an integer. Then G is critical and has chromatic number $k+1$ if and only if both G_1 and G_2 are critical and have chromatic number $k+1$.*

If G is the Hajós join of two graphs that are critical and have chromatic number $k+1$, where $k \geq 3$, then G is critical and has chromatic number $k+1$. Moreover, G has a separating set consisting of one edge and one vertex. Theorem 2.3 implies that the converse statement also holds.

Theorem 2.5 *Let G be a critical graph graph with chromatic number $k+1$ for an integer $k \geq 3$. If G has a separating set consisting of one edge and one vertex, then G is the Hajós join of two graphs.*

Next we will discuss a decomposition result for critical graphs having chromatic number $k+1$ an having an separating edge set of size k. Let G be an arbitrary graph. By an *edge cut* of G we mean a triple (X, Y, F) such that X is a non-empty proper subset of $V(G)$, $Y = V(G) \setminus X$, and $F = \partial_G(X) = \partial_G(Y)$. If (X, Y, F) is an edge cut of G, then we denote by X_F (respectively Y_F) the set of vertices of X (respectively, Y) which are incident to some edge of F. An edge cut (X, Y, F) of G is non-trivial if $|X_F| \geq 2$ and $|Y_F| \geq 2$. The following decomposition result was proved independently by T. Gallai and Toft [13].

Theorem 2.6 (Toft 1970) *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geq 3$, and let $F \subseteq E(G)$ be a separating edge set of G with $|F| \leq k$. Then $|F| = k$ and there is an edge cut (X, Y, F) of G satisfying the following properties:*

(a) *Every coloring $f \in \mathcal{CO}_k(G[X])$ satisfies $|f(X_F)| = 1$ and every coloring $f \in \mathcal{CO}_k(G[Y])$ satisfies $|f(Y_F)| = k$.*

(b) *The subgraph G_1 obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of Y_F, so that Y_F becomes a clique of G_1, is critical and has chromatic number $k+1$.*

(c) *The subgraph G_2 obtained from $G[Y]$ by adding a new vertex v and joining v to all vertices of Y_F is critical and has chromatic number $k+1$.*

A particular nice proof of this result is due to T. Gallai (oral communication to the second author). Recall that the *clique number* of a graph G, denoted by $\omega(G)$, is the largest cardinality of a clique in G. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. For the proof of the next lemma, due to Gallai, we use the fact that complements of bipartite graphs are perfect.

Lemma 2.7 *Let H be a graph and let $k \geq 3$ be an integer. Suppose that (A, B, F') is an edge cut of H such that $|F'| \leq k$ and A as well as B are cliques of H with $|A| = |B| = k$. If $\chi(H) \geq k+1$, then $|F'| = k$ and $F' = \partial_H(\{v\})$ for some vertex v of H.*

Proof. The graph H is perfect and so $\omega(H) = \chi(H) \geq k+1$. Consequently, H contains a clique X with $|X| = k+1$. Let $s = |A \cap X|$ and hence $k+1-s = |B \cap X|$. Since $|A| = |B| = k$, this implies that $s \geq 1$ and $k+1-s \geq 1$. Since X is a clique of H, the set E' of edges of H joining a vertex of $A \cap X$ with a vertex of $B \cap X$ satisfies $E' \subseteq F'$ and $|E'| = s(k+1-s)$. Clearly, $g''(s) = -2$, which implies that the function $g(s) = s(k+1-s)$ is strictly concave on the real interval $[1, k]$. Since $g(1) = g(k) = k$, we conclude that $g(s) > k$ for all $s \in (1, k)$. Since $g(s) = |E'| \leq |F'| \leq k$, this implies that $s = 1$ or $s = k$. In both cases we obtain that $|E'| = |F'| = k$, and hence $E' = F' = \partial_H(\{v\})$ for some vertex v of H. □

Based on Lemma 2.7 it is easy to give a proof of Theorem 2.6, see also the paper by Dirac, Sørensen, and Toft [7]. Theorem 2.6 is a reformulation of a result by Toft in his Ph.D thesis. Toft gave a complete characterization of the class of critical graphs, having chromatic number $k+1$ and containing a separating edge set of size k. The characterization involves critical hypergraphs.

Figure 1 shows three critical graphs with $\chi = 4$. The first graph is an odd wheel and the second graph is the Hajós join of two K_4's; both graphs belong to the class $\mathcal{C}_3$. The third graph does not belong to $\mathcal{C}_3$; it has an separating edge set of size 3, but $\lambda = 4$.

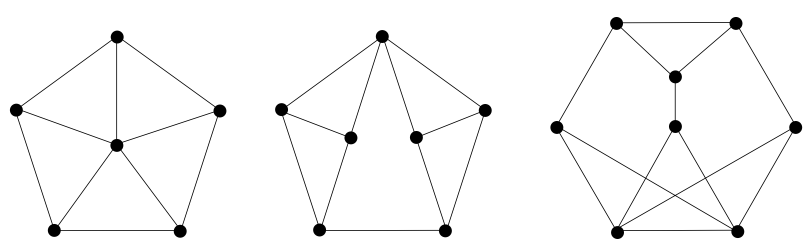

Figure 1: Three critical graphs with chromatic number $\chi = 4$.

3 Proof of the main result

Theorem 3.1 *Let $k \geq 0$ be an integer. Then the two graph classes $\mathcal{C}_k$ and $\mathcal{H}_k$ coincide.*

Proof. That the two classes $\mathcal{C}_k$ and $\mathcal{H}_k$ coincide if $0 \leq k \leq 2$ follows from Theorem 2.1(a). In this case both classes consists of all critical graphs with chromatic number $k+1$. In what follows we therefore assume that $k \geq 3$. The proof of the following claim is straightforward and left to the reader.

Claim 1 *The odd wheels belong to the class $\mathcal{C}_3$ and the complete graphs of order $k+1$ belong to the class $\mathcal{C}_k$.*

Claim 2 *Let $k \geq 3$ be an integer, and let $G = G_1 \bigtriangleup G_2$ the Hajós join of two graphs G_1 and G_2. Then G belongs to the class $\mathcal{C}_k$ if and only if both G_1, G_2 belong to the class $\mathcal{C}_k$.*

Proof: We may assume that $G = (G_1, v_1, w_1) \bigtriangleup (G_2, v_2, w_2)$ and v is the vertex of G obtained by identifying v_1 and v_2. First suppose that $G_1, G_2 \in \mathcal{C}_k$. From Theorem 2.4 it follows that G is critical and has chromatic number $k+1$. So it suffices to prove that $\lambda(G) \leq k$. To this end let u and u' be distinct vertices of G and let $p = \lambda_G(u, u')$. Then there is a system $\mathcal{P}$ of p edge disjoint u-u' paths in G. If u and u' belong both to G_1, then only one path P of $\mathcal{P}$ may contain vertices not in G_1. In this case P contains the vertex v and the edge w_1w_2. If we replace in P the subpath vPw_1 by the edge v_1w_1, we obtain a system of p edge disjoint u-u' paths in G_1, and hence $p \leq \lambda_{G_1}(u, u') \leq k$. If u and u' belong to G_2, a similar argument shows that $p \leq k$. It remains to consider the case that one vertex, say u, belongs to G_1 and the other vertex u' belongs to G_2. By symmetry we may assume that $u \neq v$. Again at most one path P of $\mathcal{P}$ uses the edge w_1w_2 and the remaining paths of $\mathcal{P}$ all uses the vertex $v(= v_1 = v_2)$. If we replace P by the path $uPw_1 + w_1v_1$, then we obtain p edge disjoint u-v_1 path in G_1, and hence $p \leq \lambda_{G_1}(u, v_1) \leq k$. This shows that $\lambda(G) \leq k$ and so $G \in \mathcal{C}_k$.

Suppose conversely that $G \in \mathcal{C}_k$. From Theorem 2.4 it follows that G_1 and G_1 are critical graphs, both with chromatic number $k+1$. So it suffices to show that $\lambda(G_i) \leq k$ for $i = 1, 2$. By symmetry it suffices to show that $\lambda(G_1) \leq k$. To this end let u and u' be distinct vertices of G_1 and let $p = \lambda_G(u, u')$. Then there is a system $\mathcal{P}$ of p edge disjoint u-u' paths in G_1. At most one path P of $\mathcal{P}$ can contain the edge v_1w_1. Clearly, there is a

v_2-w_2 path P' in G_2 not containing the edge v_2w_2. So if we replace the edge v_1w_1 of P by the path P', we get p edge disjoint u-u' paths of G, and hence $p \leq \lambda_G(u,u') \leq k$. This shows that $\lambda(G_1) \leq k$ and by symmetry $\lambda(G_2) \leq k$. Hence $G_1, G_2 \in \mathcal{C}_k$. $\triangle$

As a consequence of Claim 1 and Claim 2 and the definition of the class $\mathcal{H}_k$ we obtain the following claim.

Claim 3 *Let $k \geq 3$ be an integer. Then the class $\mathcal{H}_k$ is a subclass of $\mathcal{C}_k$.*

Claim 4 *Let $k \geq 3$ be an integer, and let G be a graph belonging to the class $\mathcal{C}_k$. If G is 3-connected, then either $k = 3$ and G is an odd wheel, or $k \geq 4$ and G is a complete graph of order $k+1$.*

Proof: The proof is by contradiction, where we consider a counterexample G whose order $|G|$ is minimum. Then $G \in \mathcal{C}_k$ is a 3-connected graph, and either $k = 3$ and G is not an odd wheel, or $k \geq 4$ and G is not a complete graph of order $k+1$. First we claim that $|G_H| \geq 2$. If $G_H = \varnothing$, then Theorem 2.2(b) implies that G is a complete graph of order $k+1$, a contradiction. If $|G_H| = 1$, then Theorem 2.2(c) implies that $k = 3$ and G is an odd wheel, a contradiction. This proves the claim that $|G_H| \geq 2$. Then let u and v be distinct high vertices of G. Since $G \in \mathcal{C}_k$, Theorem 2.1(d) implies that $\lambda_G(u,v) = k$ and, therefore, G contains a separating edge set F of size k which separates u and v. From Theorem 2.6 it then follows that there is an edge cut (X, Y, F) satisfying the three properties of that theorem. Since F separates u and v, we may assume that $u \in X$ and $v \in Y$. By Theorem2.6(a), $|Y_F| = k$ and hence each vertex of Y_F is incident to exactly one edge of F. Since Y contains the high vertex v, we conclude that $|Y_F| < |Y|$. Now we consider the graph G' obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of Y_F, so that Y_F becomes a clique of G'. By Theorem 2.6(b), G' is a critical graph with chromatic number $k+1$. Clearly, every vertex of Y_F is a low vertex of G and every vertex of X has in G' the same degree as in G. Since X contains the high vertex u of G, this implies that $|X_F| < |X|$. Since G is 3-connected, we conclude that $|X_F| \geq 3$ and that G' is 3-connected.

Now we claim that $\lambda(G') \leq k$. To prove this, let x and y be distinct vertices of G'. If x or y is a low vertex of G', then $\lambda_{G'}(x,y) \leq k$ and there is nothing to prove. So assume that both x and y are high vertices of G'. Then both vertices x and y belong to X. Let $p = \lambda_{G'}(x,y)$ and let $\mathcal{P}$ be a system of p edge disjoint x-y paths in G'. We may choose $\mathcal{P}$ such that the number of edges in $\mathcal{P}$ is minimum. Let $\mathcal{P}_1$ be the paths in $\mathcal{P}$ which uses edges of F. Since $|Y_F| = k$ and each vertex of Y_F is incident with exactly one edge of F, this implies that each path P in $\mathcal{P}_1$ contains exactly two edges of F. Since $|X_F| < |X|$ and $|Y_F| < |Y|$, there are vertices $u' \in X \setminus X_F$ and $v' \in Y \setminus Y_F$. By Theorem 2.1(d) it follows that $\lambda_G(u', v') = k$ and, therefore, there are k edge disjoint u'-v' paths in G. Since $|Y_F| = k$, for each vertex $z \in Y_F$, there is a v'-z path P_z in $G[Y]$ such that these paths are edge disjoint. Now let P be an arbitrary path in $\mathcal{P}_1$. Then P contains exactly two vertices of Y_F, say z and z', and we can replace the edge zz' of the path P by a z-z' path contained in $P_z \cup P_{z'}$. In this way we obtain a system of p edge disjoint x-y paths in G, which implies that $p \leq \lambda_G(x, y) \leq k$. This proves the claim that $\lambda(G') \leq k$. Consequently $G' \in \mathcal{C}_k$. Clearly, $|G'| < |G|$ and either $k = 3$ and G' is not an odd wheel, or $k \geq 4$ and G is not a complete graph of order $k + 1$. This, however, is a contradiction to the choice of G. Thus the claim is proved. $\triangle$

Claim 5 *Let $k \geq 3$ be an integer, and let G be a graph belonging to the class $\mathcal{C}_k$. If G has a separating vertex set of size 2, then $G = G_1 \triangle G_2$ is the Hajós sum of two graphs G_1 and G_2, which both belong to $\mathcal{C}_k$.*

Proof: If G has a separating set consisting of one edge and one vertex, then Theorem 2.5 implies that G is the Hajoś join of two graphs G_1 and G_2. By Claim 2 it then follows that both G_1 and G_2 belong to $\mathcal{C}_k$ and we are done. It remains to consider the case that G does not contain a separating set consisting of one edge and one vertex. By assumption, there is a separating vertex set of size 2, say $S = \{u, v\}$. Then Theorem 2.3 implies that $G - S$ has exactly two components H_1 and H_2 such that the graphs $G_i = G[V(H_i) \cup S]$ with $i = 1, 2$ satisfies the three properties of that theorem. In particular, we have that $G_1' = G_1 + uv$ is critical and has chromatic number k. By Theorem 2.1(c), it then follows that $\lambda_{G_1'}(u, v) \geq k$ implying that $\lambda_{G_1}(u, v) \geq k - 1$. Since $G \in \mathcal{C}_k$, we then conclude that $\lambda_{G_2}(u, v) \leq 1$. Since G_2 is connected, this implies that G_2 has a bridge e. Since $k \geq 3$, we conclude that $\{u, e\}$ or $\{v, e\}$ is a separating set of G, a contradiction. $\triangle$

As a consequence of Claim 4 and Claim 5, we conclude that the class $\mathcal{C}_k$ is a subclass of the class $\mathcal{H}_k$. Together with Claim 3 this yields $\mathcal{H}_k = \mathcal{C}_k$ as wanted. $\square$

Proof of of Theorem 1.2: For the proof of this theorem let G be a non-empty graph with $\lambda(G) = k$. By (1.1) we obtain that $\chi(G) \leq k+1$. If one block H of G belongs to $\mathcal{H}_k$, then $H \in \mathcal{C}_k$ (by Theorem 3.1) and hence $\chi(G) = k+1$ (by (1.2).

Assume conversely that $\chi(G) = k+1$. Then G contains a subgraph H which is critical and has chromatic number $k+1$. Clearly, $\lambda(H) \leq \lambda(G) \leq k$, and, therefore, $H \in \mathcal{C}_k$. By Theorem 2.1(b), H contains no separating vertex. We claim that H is a block of G. For otherwise, H would be a proper subgraph of a block G' of G. This implies that there are distinct vertices u and v in H which are joined by a path P of G with $E(P) \cap E(H) = \varnothing$. Since $\lambda_H(u,v) \geq k$ (by Theorem 2.1(c)), this implies that $\lambda_G(u,v) \geq k+1$, which is impossible. This proves the claim that H is a block of G. By Theorem 3.1, $\mathcal{C}_k = \mathcal{H}_k$ implying that $H \in \mathcal{H}_k$. This completes the proof of the theorem $\square$

The case $\lambda = 3$ of Theorem 1.2 was obtained earlier by Alboulker *et al.* [1]; their proof is similar to our proof. Let $\mathcal{L}_k$ denote the class of graphs G satisfying $\lambda(G) \leq k$. It is well known that membership in $\mathcal{L}_k$ can be tested in polynomial time. It is also easy to show that there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, decides whether G or one of its blocks belong to $\mathcal{H}_k$. So it can be tested in polynomial time whether a graph $G \in \mathcal{L}_k$ satisfies $\chi(G) \leq k$. Moreover, the proof of Theorem 1.2 yields a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring of $\mathcal{CO}_k(G)$ when such a coloring exists. This result provides a positive answer to a conjecture made by Alboulker *et al.* [1, Conjecture 1.8]. The case $k = 3$ was solved by Alboulker *et al.* [1].

Theorem 3.2 *For fixed $k \geq 1$, there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring in $\mathcal{CO}_k(G)$ or a block belonging to $\mathcal{H}_k$.*

Sketch of Proof: The Theorem is evident if $k = 1, 2$; and the case $k = 3$ was solved by Alboulker *et al.* [1]. Hence we assume that $k \geq 4$ and $G \in \mathcal{L}_k$. If we find for each block H of G a coloring in $\mathcal{CO}_k(H)$, we can piece these colorings together by permuting colors to obtain a coloring in $\mathcal{CO}_k(G)$. Hence we may assume that G is a block. First, we check whether G has a separating set S consisting of one vertex and one edge. If we find such a set, say $S = \{v, e\}$ with $v \in V(G)$ and $e \in E(G)$. Then $G - e$ is the union of two connected graphs G_1 and G_2 having only vertex v in common where $e = w_1w_2$ and $w_i \in V(G_i)$ for $i = 1, 2$. Both blocks $G_1' = G_1 + vw_1$ and $G_2' = G_2 + vw_2$

belong to $\mathcal{L}_k$. Now we check whether these blocks belong to $\mathcal{H}_k$. If both blocks G'_1 and G'_2 belong to $\mathcal{H}_k$, then $vw_i \notin E(G_i)$ for $i = 1, 2$, and hence G belongs to $\mathcal{H}_k$ and we are done. If one of the blocks, say G'_1 does not belong to $\mathcal{H}_k$, we can construct a coloring $f_1 \in \mathcal{CO}_k(G'_1)$. Moreover, no block of G_2 belongs to $\mathcal{H}_k$, hence we can construct a coloring $f_2 \in \mathcal{CO}_k(G_2)$. Then $f_1 \in \mathcal{CO}_k(G_1)$ and $f_1(v) \neq f_1(w_1)$. Since $k \geq 4$, we can permute colors in f_2 such that $f_1(v) = f_2(v)$ and $f_1(w_1) \neq f_2(w_2)$. Consequently, $f = f_1 \cup f_2$ belongs to $\mathcal{CO}_k(G)$ and we are done.

It remains to consider the case that G contains no separating set consisting of one vertex and one edge. Then let p denote the number of vertices of G whose degree is greater that k. If $p \leq 1$, then let v be a vertex of maximum degree in G. Color v with color 1 and let L be a list assignment for $H = G - v$ satisfying $L(u) = \{2, 3, \ldots, k\}$ if $vu \in E(G)$ and $L(u) = \{1, 2, \ldots, k\}$ otherwise. Then H is connected and $|L(u)| \geq d_H(u)$ for all $u \in V(H)$. Now we can use the degree version of Brooks' theorem, see [12, Theorem 2.1]. Either we find a coloring f of H such that $f(u) \in L(u)$ for all $u \in V(H)$, yielding a coloring of $\mathcal{CO}_k(G)$, or $|L(u)| = d_H(u)$ for all $u \in V(H)$ and each block of H is a complete graph or an odd cycle. In this case, $d_H(u) \in \{k, k-1\}$ for all $u \in V(H)$ and, since $k \geq 4$, each block of H is a K_k or a K_2. Since G contains no separating set consisting of one vertex and one edge, this implies that $H = K_k$ and so $G = K_{k+1} \in \mathcal{H}_k$ and we are done. If $p \geq 2$, then we choose two vertices u and u' whose degrees are greater that k. Then we construct an edge cut (X, Y, F) with $u \in X$, $u' \in Y$, and $|F| = \lambda_G(u, u')$. We may assume that $a = |X_F|$ and $b = |Y_F|$ satisfies $a \leq b \leq k$. If $b \leq k-1$, then both graphs $G[X]$ and $G[Y]$ belong to $\mathcal{L}_k$ and there are colorings $f_X \in \mathcal{CO}_k(G[X])$ and $f_Y \in \mathcal{CO}_k(G[Y])$. Note that no block of these two graphs can belong to $\mathcal{H}_k$. By permuting colors in f_Y, we can combine the two colorings f_X and f_Y to obtain a coloring $f \in \mathcal{CO}_k(G)$ (by Lemma 2.7). If $a < b = k$, then we consider the graph G_1 obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of Y_F, so that Y_F becomes a clique of G_1. Then G_1 belongs to $\mathcal{L}_k$ (see the proof of Claim 4) and, since G contains no separating set consisting of one vertex and one edge, the block G_1 does not belongs to $\mathcal{H}_k$. Hence there are colorings $f_1 \in \mathcal{CO}_k(G_1)$ and $f_Y \in \mathcal{CO}_k(G[Y])$. Then the restriction of f_1 to X yields a coloring $f_X \in \mathcal{CO}_k(G[X])$ such that $|f_X(X)| \geq 2$. By permuting colors in f_Y, we can combine the two colorings f_X and f_Y to obtain a coloring $f \in \mathcal{CO}_k(G)$ (by Lemma 2.7). It remains to consider the case $a = b = k$. Then let G_2 be the graph obtained from $G[Y \cup X_F]$ by adding all edges between the vertices of X_F, so that X_F becomes a clique of G_2. Then we find colorings $f_1 \in \mathcal{CO}_k(G_1)$ and $f_2 \in \mathcal{CO}_k(G_2)$ and, hence, colorings $f_X \in \mathcal{CO}_k(G[X])$ and $f_Y \in \mathcal{CO}_k(G[Y])$ such that $|f_X(X)| \geq 2$ and $|f_Y(Y)| \geq 2$. By permuting colors in f_Y, we can combine the two colorings f_X and f_Y to obtain a coloring $f \in \mathcal{CO}_k(G)$ (by Lemma 2.7). $\square$

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